Assignment 3

Coverage: 15.4 in Text.

Exercises: 15.4. no 13, 15, 17, 19, 22, 24, 25, 27, 29, 30, 34, 35, 40, 41, 42, 43, 46.

Submit no. 22, 24, 27, 35, 46 by Oct 4.

29. Find the area of one leaf of the rose $r = 12\cos 3\theta$.

Solution. As the cosine function is 2π-periodic, $\cos 3\theta$ is $2\pi/3$-periodic. It suffices to plot its graph in $[-\pi/3, \pi/3]$. Observing that in this interval, $\cos 3\theta$ is non-negative only on $[-\pi/6, \pi/6]$, there is one leaf sitting in $[-\pi/6, \pi/6]$. By rotating it by $2\pi/3$ and then by $4\pi/3$, we obtain the full graph of the rose which consists of three identical leaves.

By symmetry, the area of one leaf is

$$2\int_0^{\pi/6}\int_0^{12\cos 3\theta} dr d\theta = 8 \ .$$

41. In this problem we establish the famous formula by using double integral in a tricky way. Setting

$$a = \int_{-\infty}^{\infty} e^{-x^2}\, dx \ .$$

We have

$$\begin{aligned} a^2 &= \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{y^2}\, dy \\ &= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA(x,y) \\ &= \lim_{R\to\infty} \iint_{D_R} e^{-x^2-y^2}\, dA(x,y) \\ &= \lim_{R\to\infty} \int_0^{2\pi}\int_0^R e^{-r^2} r\, dr d\theta \\ &= \lim_{R\to\infty} \int_0^{2\pi}\int_0^{R^2} e^{-s}\, ds d\theta \\ &= \pi. \end{aligned}$$

Hence

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \ .$$

Supplementary Problems

1. Express the hyperbola $x^2 - y^2 = 1$ $(y \geq 0)$ in polar coordinates. What is the range of θ?
2. Let D be the sector bounded by the line $y = ax, a > 0$, the positive y-axis and the circle $x^2 + y^2 = r^2$. Use cartesian coordinates in your integration to show that its area is given by $r^2\Theta/2$ where Θ is the angle between $y = ax$ and the y-axis.
3. Let D be the region bounded by the graph of $y = \sqrt{1-x^2}+1$ and the x-axis over $0 \leq x \leq 1$. Describe it in polar coordinates.